Finite Math - J-term 2019

Lecture Notes - 1/7/2019

Homework

- Section 2.5 - 1, 3, 5, 31, 34, 67, 68

- Section 2.6 - 13, 16, 18, 26, 27, 30, 32, 61, 63, 65, 68

Section 2.5 - Exponential Functions

Definition 1 (Exponential Function)**.** *An* exponential function *is a function of the form*

$$f(x) = b^x, \;\; b > 0, \;\; b \neq 1.$$

b is called the base*.*

Why the restrictions on b?

- If $b = 1$, then $f(x) = 1^x = 1$ for all x values. Not a very interesting function!

- As an example of the case when $b < 0$, suppose $b = -1$. Then

$$f\left(\frac{1}{2}\right) = (-1)^{1/2} = \sqrt{-1} = i$$

  an imaginary number! This kind of thing will always happen if b is negative.

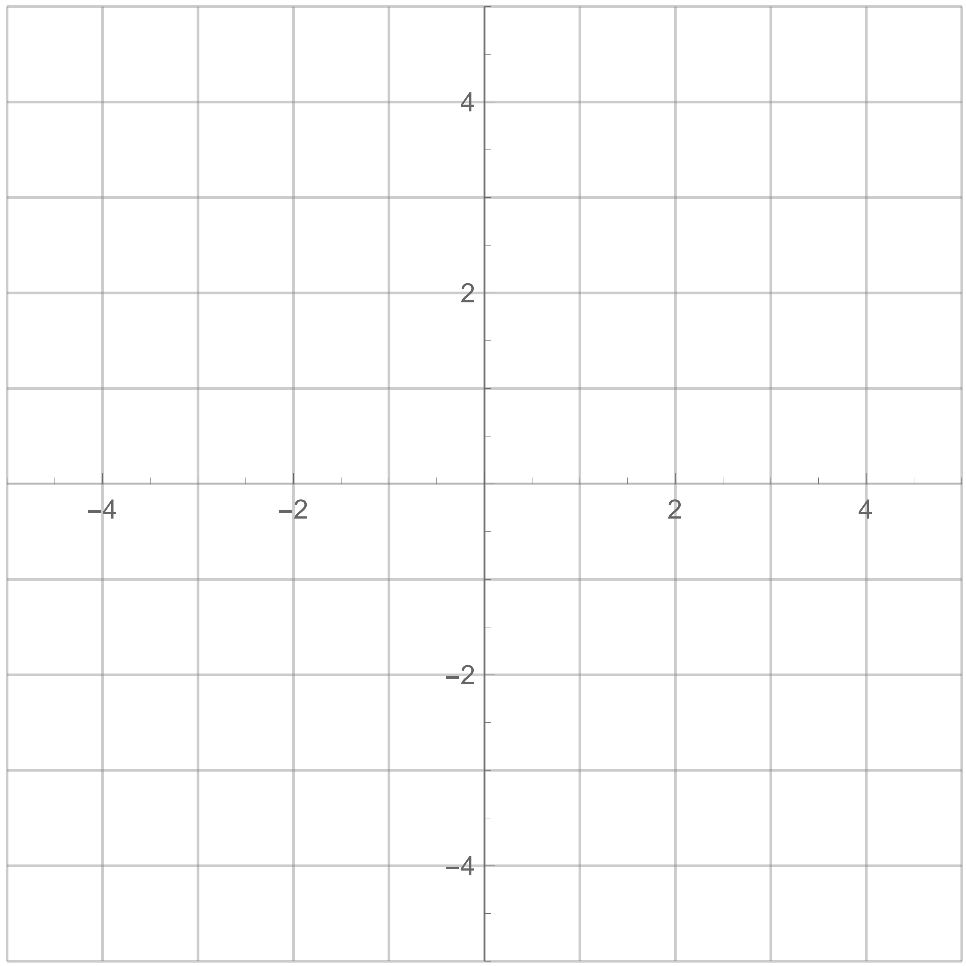
- If $b = 0$, then for negative x values, f is not defined. For example,

$$f(-1) = 0^{-1} = \frac{1}{0} = undefined.$$

Let's get an idea of what these functions look like by graphing a few of them.

Example 1. *Sketch the graph of $f(x) = 2^x$.*

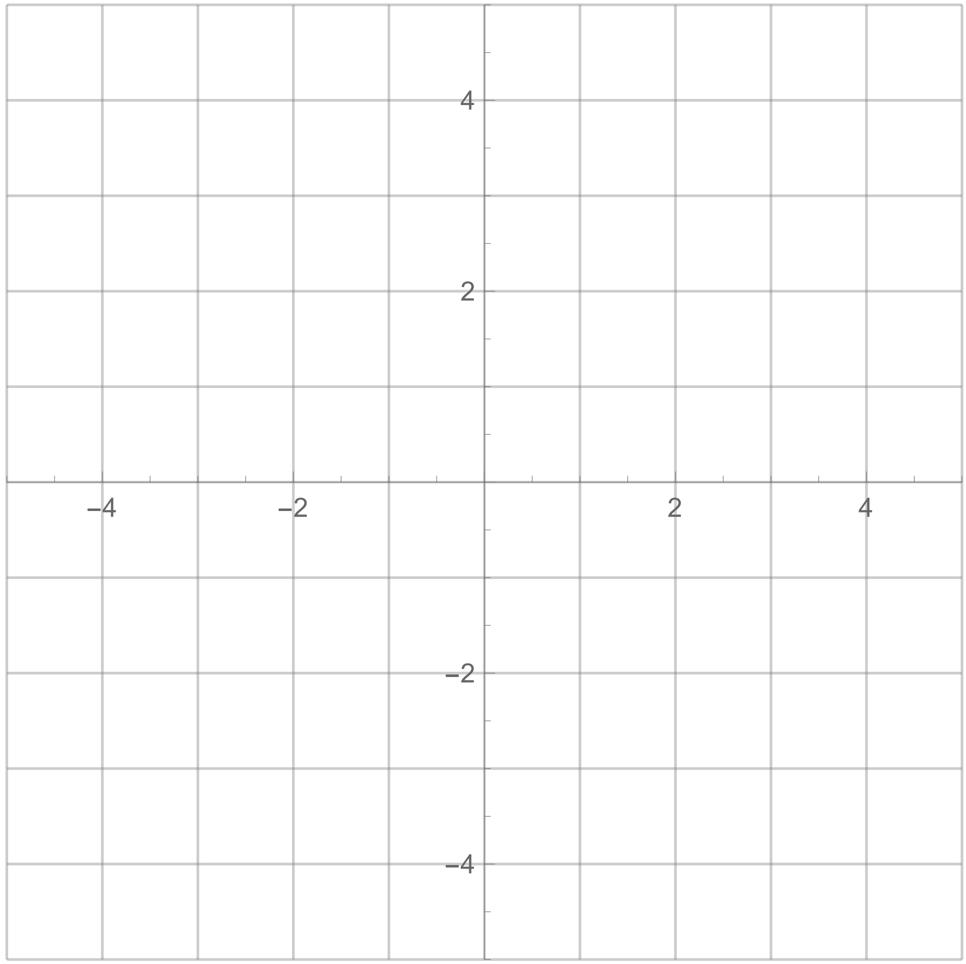
Solution.



When $b > 1$, the graph of $f(x) = b^x$ has the same basic shape as 2^x, but may be steeper or more gradual. Let's see what happens when $b < 1$.

Example 2. *Sketch the graph of* $f(x) = \left(\frac{1}{2}\right)^x$.

Solution.



Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

Properties of Exponential Functions.

Property 1 (Graphical Properties of Exponential Functions)**.** *The graph of* $f(x) = b^x$*,* $b > 0$*,* $b \neq 1$ *satisfies the following properties:*

(1) *All graphs pass through the point* $(0, 1)$*.*

(2) *All graphs are continuous.*

(3) *The* x*-axis is a horizontal asymptote.*

(4) b^x *is increasing if* $b > 1$.

(5) b^x *is decreasing if* $0 < b < 1$.

Property 2 (General Properties of Exponents)**.** *Let* $a, b > 0$, $a, b \neq 1$, *and* x, y *be real numbers. The following properties are satisfied:*

(1) $a^x a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

(2) $a^x = a^y$ *if and only if* $x = y$

(3) $a^x = b^x$ *for all* x *if and only if* $a = b$

A Special Number: e**.** There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which the quantity

$$\left(1 + \frac{1}{x}\right)^x$$

approaches as x tends towards ∞.

This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Example 3. *In 2013, the estimated world population was* 7.1 *billion people with a relative growth rate of* 1.1%*.*

(a) *Write a function modeling the world population* t *years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Solution.

Example 4. *The population of some countries has a relative growth rate of* 3% *per year. Suppose the population of such a country in 2012 is* 6.6 *million.*

(a) *Write a function modeling the population t years after 2012.*

(b) *What is the expected population in 2018? 2022?*

Solution.

Section 2.6 - Logarithmic Functions

Before we can accurately talk about what logarithms are, let's first remind ourselves about inverse functions.

Inverse Functions. The inverse of a function is given by running the function backwards. But when can we do this?

Consider the function $f(x) = x^2$. If we run f backwards on the value 1, what x-value do we get?

Since $(1)^2 = 1$ and $(-1)^2 = 1$, we get *two* values when we run x^2 backward! So x^2 is not invertible.

This shows that not every function is invertible. To get the inverse of a function, we need each range value to come from *exactly one* domain value. We call such functions *one-to-one*.

If we have a one-to-one function

$$y = f(x)$$

we can form the *inverse function* by switching x and y and solving for y:

$$x = f(y) \xrightarrow{\text{solve for } y} y = f^{-1}(x).$$

Logarithms. We will focus on one particular inverse function: the inverse of the function $f(x) = b^x$ ($b > 0$, $b \neq 1$).

Definition 2 (Logarithm)**.** *The* logarithm of base b *is defined as the inverse of* b^x*. That is,*

$$y = b^x \iff x = \log_b y.$$

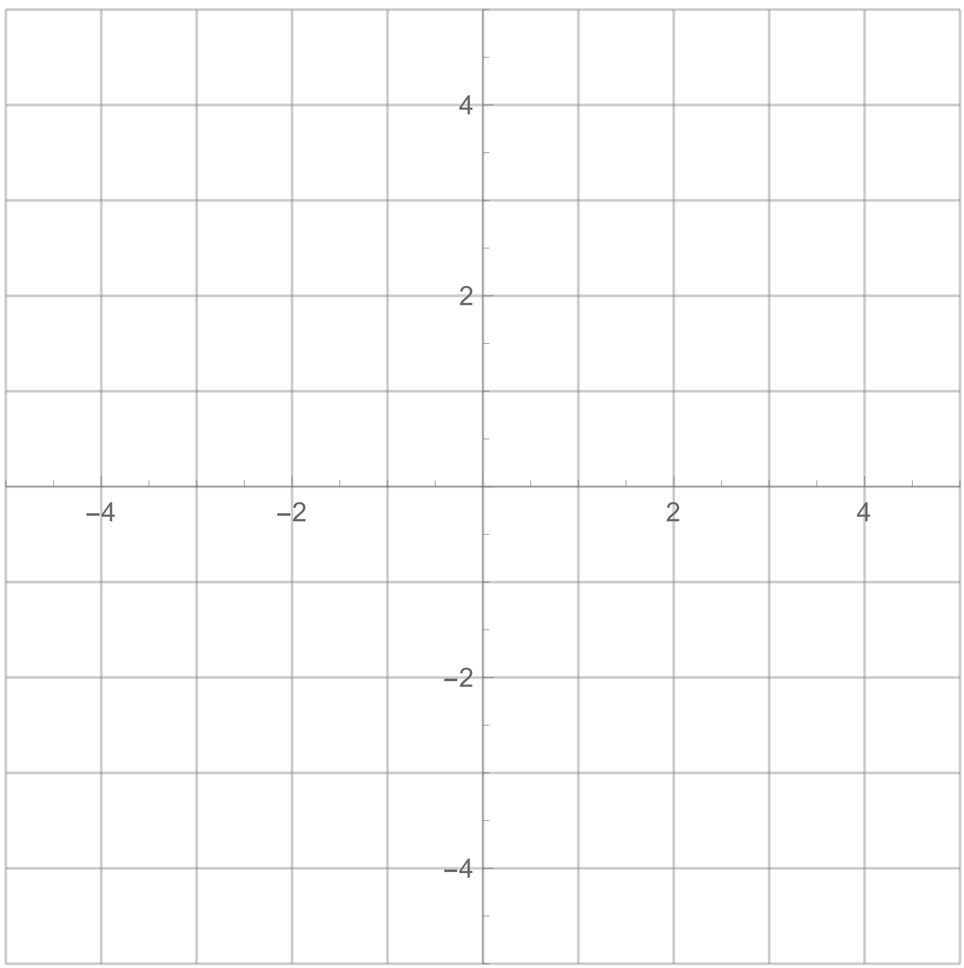
Since the domain and range switch when we take inverses, we have

function	domain	range
$f(x) = b^x$	$(-\infty, \infty)$	$(0, \infty)$
$f(x) = \log_b x$	$(0, \infty)$	$(-\infty, \infty)$

Let's look at one example of a graph of a logarithmic function.

Example 5. *Sketch the graph of* $f(x) = \log_2 x$*.*

Solution.



Properties of Logarithms. Since logarithms are inverse to exponential functions, we get some convenient properties for logarithms:

Property 3 (Properties of Logarithms)**.** *Let* $b, M, N > 0$*,* $b \neq 1$*, and* p, x *be real numbers. Then*

(1) $\log_b 1 = 0$

(2) $\log_b b = 1$

(3) $\log_b b^x = x$

(4) $b^{\log_b x} = x$

(5) $\log_b MN = \log_b M + \log_b N$

(6) $\log_b \dfrac{M}{N} = \log_b M - \log_b N$

(7) $\log_b M^p = p \log_b M$

(8) $\log_b M = \log_b N$ *if and only if* $M = N$

Properties 3 and 7 above are incredibly important to us as we will use them frequently in the study of financial mathematics! Learn these properties well!!

The Natural Logarithm. Just as with exponential functions, if we choose our base to be the number e, we get a special logarithm, the *natural logarithm.*

$$\log_e x = \ln x.$$

We can actually rewrite a logarithm in any base in terms of ln:

$$\log_b x = \frac{\ln x}{\ln b}$$

(See the textbook for a proof of this.)

Using Properties of Logarithms and Exponents.

Example 6. *Solve for x in the following equations:*

(a) $7 = 2e^{0.2x}$

(b) $16 = 5^{3x}$

(c) $8000 = (x - 4)^3$

Solution.

A quick reminder of different types of exponents:

- $a^{-n} = \frac{1}{a^n}$
- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $a^{1/2} = \sqrt{a}$
  - $a^{1/3} = \sqrt[3]{a}$
- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

Example 7. *Solve for x in the following equations:*

(a) $75 = 25e^{-x}$

(b) $42 = 7^{2x+3}$

(c) $200 = (2x-1)^5$

Solution.

Applications. Recall that exponential growth/decay models are of the form

$$A = ce^{rt}.$$

Using the natural logarithm, we can solve for the rate of growth/decay, r, and the time elapsed, t. Let's see this in an example.

Example 8. *The isotope carbon-*14 *has a half-life (the time it takes for the isotope to decay to half of its original mass) of* 5730 *years.*

(a) *At what rate does carbon-*14 *decay?*

(b) *How long would it take for* 90% *of a chunk of carbon-*14 *to decay?*

Solution.